

# STABILITY AND CONVERGENCE OF THE SASAKI-RICCI FLOW

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**ABSTRACT.** We introduce a holomorphic sheaf  $\mathcal{E}$  on a Sasaki manifold  $S$  and study two new notions of stability for  $\mathcal{E}$  along the Sasaki-Ricci flow related to the ‘jumping up’ of the number of global holomorphic sections of  $\mathcal{E}$  at infinity. First, we show that if the Mabuchi K-energy is bounded below, the transverse Riemann tensor is bounded in  $C^0$  along the flow, and the  $C^\infty$  closure of the Sasaki structure on  $S$  under the diffeomorphism group does not contain a Sasaki structure with strictly more global holomorphic sections of  $\mathcal{E}$ , then the Sasaki-Ricci flow converges exponentially fast to a Sasaki-Einstein metric. Secondly, we show that if the Futaki invariant vanishes, and the lowest positive eigenvalue of the  $\bar{\partial}$  Laplacian on global sections of  $\mathcal{E}$  is bounded away from zero uniformly along the flow, then the Sasaki-Ricci flow converges exponentially fast to a Sasaki-Einstein metric.

## 1. INTRODUCTION

Sasaki geometry is a generalization of Kähler geometry with applications to the AdS/CFT correspondence in theoretical physics. It is an important problem to determine when Sasaki-Einstein metrics exist. When the basic first Chern class is non-positive, the existence theory is well developed and generalizes the results of Aubin and Yau [1, 40]. However, when the basic first Chern class is positive, there are known obstructions to the existence of Sasaki-Einstein metrics; see, for example, the work of Futaki, Ono, and Wang [15], and Gauntlett, Martelli, Sparks and Yau [16]. It is expected that a suitably modified version of the famous conjecture of Yau [41] should hold in the Sasaki setting; that is, existence of Sasaki-Einstein metrics with positive basic first Chern class should be equivalent to some geometric invariant theory notion of stability. Recently, a flow approach to the Sasaki-Einstein problem was developed by Smoczyk, Wang and Zhang in [33], where they introduced the Sasaki-Ricci flow which generalizes the Kähler-Ricci flow, and extended the results of Cao [7]. There is now a large body of work relating various notions of algebraic stability to convergence of the Kähler-Ricci flow; see for example [26, 35, 39] and the references therein. It is desirable to determine whether analogues of these results hold in the Sasaki case. A particular form of stability which arises in the Kähler setting concerns the degeneration of eigenvalues of various Laplacians along the flow. Phong and Sturm [27], and Phong, Song, Sturm and Weinkove [28] proved convergence of the Kähler-Ricci flow assuming a bound below for the Mabuchi functional and stability conditions for the lowest positive eigenvalue of the  $\bar{\partial}$  Laplacian on  $T^{1,0}$  vector fields (cf. condition (B) in [27], and condition (S) in [28]). In [42], Zhang proved convergence of the flow under a non-degeneracy condition for the ‘second’ eigenvalue of a modified Laplacian on smooth functions.

In the Sasaki case, it is natural to ask whether forms of stability analogous to those studied in [27, 28, 42] are available, and whether they imply the convergence of the Sasaki-Ricci flow to a transverse Kähler-Einstein metric when the Futaki invariant vanishes or the Mabuchi functional is bounded below. We aim to address these questions presently. We would like to point out a few simple observations which hint at the difficulties ahead. Assume for simplicity that the Sasaki structure is regular, so that the Sasaki manifold  $S$  is diffeomorphic to a Kähler manifold  $S/U(1)$  with a principle  $U(1)$  bundle. In this case, the stability conditions we seek are necessarily the pull back of the Kähler stability conditions under the quotient map  $\pi : S \rightarrow S/U(1)$ . The first observation is that pulling back sections of the tangent bundle by  $\pi$  does not yield a module over the ring of smooth functions. Thus, if we seek to generalize condition (B) of [27] or condition (S) of [28] we must work in the realm of locally free sheaves of modules over the ring of *basic* functions. More generally, when the Sasaki

structure is irregular, so that the leaf space of the Reeb foliation does not have the structure of a Kähler orbifold, how do we identify the space of “holomorphic vector fields”? A general approach to this problem is to extend Kähler notions of stability to Kähler orbifolds and then formulate some related notion of stability which behaves well under approximation by quasi-regular Sasaki structures. We prefer the point of view which avoids these approximation techniques. In [6, 15, 24], the Lie algebra of holomorphic Hamiltonian vector fields was identified as a central object of study in the existence of Sasaki-Einstein and extremal Sasaki metrics. Is it possible to view these vector fields as the kernel of a  $\bar{\partial}$  operator on the global sections of some sheaf  $\mathcal{E}$ ? More importantly, if such an  $\mathcal{E}$  exists, can we relate the convergence of the Sasaki-Ricci flow to the eigenvalues of the  $\bar{\partial}$  Laplacian on the global sections of  $\mathcal{E}$ ? In this paper we answer these questions in the affirmative. We identify a sheaf  $\mathcal{E}$ , called the sheaf of transverse foliate vector fields, which has a well defined  $\bar{\partial}$  operator, with the property that the global holomorphic sections of  $\mathcal{E}$  correspond precisely to the Hamiltonian holomorphic vector fields. We consider the following notions of stability;

- (M) The Mabuchi energy is bounded below.
- (F) The Futaki invariant vanishes.
- (C) Let  $(\xi, \eta, \Phi)$  be a Sasaki structure on  $S$ . Then the  $C^\infty$  closure of the orbit of the triple  $(\xi, \eta, \Phi)$  under the diffeomorphism group of  $S$  does not contain any Sasaki structure  $(\xi_\infty, \eta_\infty, \Phi_\infty)$  with the property that the dimension of the space of global holomorphic sections of the sheaf of transverse foliate vector fields with respect to  $(\xi_\infty, \eta_\infty, \Phi_\infty)$  has dimension strictly higher than the dimension of the space of global holomorphic sections of the sheaf of transverse foliate vector fields with respect to  $(\xi, \eta, \Phi)$ .

Condition (C) generalizes condition (B) of [27]. In light of Proposition 5.4 below, condition (F) is at least *a priori* weaker than condition (M). We refer the reader to §5 for details on the sheaf  $\mathcal{E}$ , and its holomorphic structure. Our first theorem extends Theorem 1 in [27].

**Theorem 1.1.** *Let  $(S, \xi, \eta, \Phi, g_0)$  be a compact Sasaki manifold with  $c_1^B(S) > 0$ . Assume that  $g(t)$  is a solution of the Sasaki-Ricci flow with  $g(0) = g_0$ , and  $(2n+2)g_0$  is in the basic first Chern class of  $(S, \xi, \eta, \Phi, g_0)$ . Assume that the transverse Riemann curvature is bounded along the flow.*

- (i) *If condition (M) holds, then we have for any  $s \geq 0$*

$$\lim_{t \rightarrow \infty} \|R^T(t) - (2n+2)g^T(t)\|_{(s)} = 0$$

*where  $\|\cdot\|_{(s)}$  denotes the Sobolev norm of order  $s$  with respect to the metric  $g(t)$ .*

- (ii) *If both conditions (M) and (C) hold, then the Sasaki-Ricci flow converges exponentially fast in  $C^\infty$  to a Sasaki-Einstein metric.*

We remove the condition on the boundedness of  $Rm^T$  by introducing the stability condition (T), which generalizes condition (S) of [28]. The following theorem extends the results of [28], and [42].

**Theorem 1.2.** *Let  $(S, \xi, \eta, \Phi, g_0)$  be a compact Sasaki manifold with  $c_1^B(S) > 0$ . Assume that  $g(t)$  is a solution of the Sasaki-Ricci flow with  $g(0) = g_0$ , and  $(2n+2)g_0 \in c_1^B(S)$ . Let  $\lambda_t$  be the lowest strictly positive eigenvalue of the Laplacian  $\square_{\mathcal{E}} := -(g^T)^{j\bar{k}} \nabla_j^T \nabla_{\bar{k}}^T$  acting on smooth global sections of  $\mathcal{E}^{1,0}$ .*

- (i) *If condition (F) and condition*

$$(T) \inf_{t \in [0, \infty)} \lambda_t > 0$$

*hold, then the metrics  $g(t)$  converge exponentially fast in  $C^\infty$  to a Sasaki-Einstein metric.*

- (ii) *Conversely, if the metrics  $g(t)$  converge in  $C^\infty$  to a Sasaki-Einstein metric, then conditions (F) and (T) hold.*
- (iii) *In particular, if the metrics  $g(t)$  converge in  $C^\infty$  to a Sasaki-Einstein metric, then they converge exponentially fast in  $C^\infty$  to this metric.*

The condition “ $(2n+2)g_0$  is in the basic first Chern class of  $(S, \xi, \eta, \Phi, g_0)$ ” in the above Theorems is not restrictive in light of the so-called “ $\mathcal{D}$ - homothetic transformations” introduced by Tanno [36]; see §3. The outline of this paper is as follows; in §2 we provide an introduction to Sasaki geometry. In §3 we discuss perturbations of Sasaki structures and the Sasaki-Ricci flow. We present an argument for a specific choice of the initial value of the transverse Kähler potential and point out a some consequences of this normalization. We also present previous results on the Sasaki-Ricci flow which we need. In §4 we extend the well known estimates of Hamilton [19] and Shi [32] for the Ricci flow to the Sasaki-Ricci flow on compact Sasaki manifolds. In particular, we prove

**Theorem 1.3.** *Let  $(S, g_0)$  be a compact Sasaki manifold of dimension  $2n+1$ , and suppose that  $g(t)$  is a solution of the normalized Sasaki-Ricci flow, with  $g(0) = g_0$ . Then, for each  $\alpha > 0$ , and every  $m \in \mathbb{N}$ , there exists a constant  $C_m$  depending only on  $m, n$  and  $\max\{\alpha, 1\}$  such that if  $K$  satisfies*

$$|Rm^T(x, t)|_{g^T(x, t)} \leq K \text{ for every } x \in S, \text{ and } t \in [0, \frac{\alpha}{K}]$$

*then, the bound*

$$\max \left\{ |\nabla^m Rm(x, t)|_{g(t)}, |\nabla^m Rm^T(x, t)|_{g^T(x, t)} \right\} \leq \frac{C_m \max\{K^{1/2}, K\}}{t^{m/2}}$$

*holds for every  $x \in S$  and  $t \in (0, \frac{\alpha}{K}]$ .*

In §5 we take up stability on Sasaki manifolds. We begin by discussing the Futaki invariant and the Mabuchi energy, extending some well known results from the Kähler theory to the Sasaki setting. We then construct the sheaf  $\mathcal{E}$  and discuss its properties. In §6 we prove Theorem 1.1 part (i), and in §7 we reduce the proof of Theorem 1.1 part (ii) to obtaining a positive lower bound for the smallest positive eigenvalue of the  $\bar{\partial}$  Laplacian on the global sections of  $\mathcal{E}^{1,0}$ . In §8 we complete the proof of Theorem 1.1 by discussing some compactness results for Sasaki manifolds, and showing that condition (C) implies a positive lower bound for the smallest positive eigenvalue of the  $\bar{\partial}$  Laplacian. In §9 we prove Theorem 1.2.

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## 2. SASAKI MANIFOLDS AND TRANSVERSE KÄHLER GEOMETRY

In this section we give a brief introduction to Sasaki geometry. For a more thorough introduction we refer the reader to [4, 10, 34]. A Sasakian manifold of dimension  $2n+1$  is a Riemannian manifold  $(S^{2n+1}, g)$  with the property that its metric cone  $(C(S) = \mathbb{R}_{>0} \times S, \bar{g} = dr^2 + r^2g)$  is Kähler. A great deal of the geometry of Sasaki manifolds is induced by the *Euler* vector field  $r\partial_r$ . One can easily show that  $r\partial_r$  is real holomorphic, that is,  $\mathcal{L}_{r\partial_r}J = 0$ . A particularly important role in Sasaki geometry is played by the Reeb vector field, which is naturally induced from the vector field  $r\partial_r$ .

**Definition 2.1.** *The Reeb vector field is  $\xi = J(r\partial_r)$ , where  $J$  denotes the integrable complex structure on  $C(S)$ .*

Again, the Reeb field  $\xi$  satisfies  $\mathcal{L}_\xi J = 0$ , and so  $\xi$  is real holomorphic. The restriction of  $\xi$  to the slice  $\{r = 1\}$  is a unit length Killing vector field, and its orbits thus define a one-dimensional foliation of  $S$  by geodesics called the Reeb foliation. Let  $L_\xi$  be the line bundle spanned by the non-vanishing vector field  $\xi$ . The contact subbundle  $D \subset TS$  is defined as  $D = \ker \eta$  where  $\eta(X) := g(\xi, X)$ . We have the exact sequence

$$(1) \quad 0 \rightarrow L_\xi \rightarrow TS \xrightarrow{p} Q \rightarrow 0.$$

The Sasakian metric  $g$  gives an orthogonal splitting of this sequence  $\sigma : Q \rightarrow D$  so that we identify  $Q \cong D$ , and  $TS = D \oplus L_\xi$ . Define a section  $\Phi \in \text{End}(TS)$  via the equation  $\Phi(X) = \nabla_X \xi$ . One can then check that  $\Phi|_D = J|_D$  and  $\Phi|_{L_\xi} = 0$ , and that

$$\Phi^2 = -1 + \eta \otimes \xi, \text{ and } g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields  $X$  and  $Y$  on  $S$ . In particular,  $g|_D$  is a Hermitian metric on  $D$ .

The identification  $Q \cong D$  endows the quotient bundle  $Q$  with a transverse metric  $g^T$ . There is a unique, torsion-free connection on  $Q$ , which is compatible with the metric  $g^T$ . This connection is defined by

$$\nabla_X^T V = \begin{cases} (\nabla_X \sigma(V))^p, & \text{if } X \text{ is a section of } D \\ [\xi, \sigma(V)]^p, & \text{if } X = \xi \end{cases}$$

where  $\nabla$  is the Levi-Civita connection on  $(S, g)$ ,  $\sigma$  is the splitting map induced by  $g$ , and  $p : TS \rightarrow Q$  is the projection. This connection is called the *transverse Levi-Civita connection* as it is compatible with  $g^T$ , and torsion free with respect to the bracket induced on  $Q$ . We will denote by  $Rm^T$  the curvature operator defined by the connection  $\nabla^T$ ,  $Ric^T$ , and  $R^T$  will denote the transverse Ricci curvature of scalar curvature respectively. The geometry of the manifold  $S$  is largely controlled by its transverse geometry. For local sections  $X, Y, Z, W$  of  $D$ , the curvature of  $(S, g)$  is related to the curvature of  $(Q, g^T)$  by

$$(2) \quad \begin{aligned} Rm(X, Y, Z, W) = & Rm^T(X, Y, Z, W) + g(\Phi(X), Z)g(\Phi(Y), W) \\ & - g(\Phi(X), W)g(\Phi(Y), Z) + 2g(\Phi(X), Y)g(\nabla_\xi X, W). \end{aligned}$$

The geometry orthogonal to the distribution  $D$  is uniform in the sense that for any vector fields  $X, Y \in TS$  there holds

$$(3) \quad Rm(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$(4) \quad Rm(X, \xi)Y = \eta(Y)X - g(X, Y)\xi.$$

We refer the reader to [4, 15] for more on these standard formulae.

**Definition 2.2.** For  $x \in S$ , let  $\text{orb}_\xi x$  denote the orbit of  $x$  under the action generated by the Reeb field. We define the transverse distance function  $d^T : S \times S \rightarrow \mathbb{R}$  by

$$d^T(x, y) = \inf_{\{p \in \text{orb}_\xi x, q \in \text{orb}_\xi y\}} \text{dist}(p, q)$$

The transverse distance function played a crucial role in [10]. With these preparations, we see it fitting to make a brief digression on coordinates. One can choose local coordinates  $(x, z^1, \dots, z^n)$  on a small neighbourhood  $U$  of  $p$  such that

- $\xi = \frac{\partial}{\partial x}$
- $\eta = dx + \sqrt{-1} \sum_{j=1}^n h_j dz^j - \sqrt{-1} \sum_{j=1}^n h_{\bar{j}} d\bar{z}^j$
- $\Phi = \sqrt{-1} \left\{ \sum_{j=1}^n \left( \frac{\partial}{\partial z^j} - \sqrt{-1} h_j \frac{\partial}{\partial x} \right) \otimes dz^j - \sum_{j=1}^n \left( \frac{\partial}{\partial \bar{z}^j} + \sqrt{-1} h_{\bar{j}} \frac{\partial}{\partial x} \right) \otimes d\bar{z}^j \right\}$
- $g = \eta \otimes \eta + 2 \sum_{j,l=1}^n h_{j\bar{l}} dz^j d\bar{z}^l$
- $D \otimes \mathbb{C}$  is spanned by

$$X_j := \frac{\partial}{\partial z^j} - \sqrt{-1} h_j \frac{\partial}{\partial x} \quad X_{\bar{j}} := \frac{\partial}{\partial \bar{z}^j} + \sqrt{-1} h_{\bar{j}} \frac{\partial}{\partial x}$$

where  $h : U \rightarrow \mathbb{R}$  is a local, basic function (ie.  $\frac{\partial}{\partial x} h = 0$ ), and we have used  $h_j = \frac{\partial}{\partial z^j} h$ , and  $h_{j\bar{l}} = \frac{\partial^2}{\partial z^j \partial \bar{z}^l} h$ . See, for example, [17]. We can additionally assume that in these coordinates  $h_j(p) = 0$ . An important observation is that the transverse Christoffel symbols with mixed barred and unbarred indices are identically zero. Moreover, the pure barred and unbarred Christoffel symbols are given by the familiar formula from Kähler geometry

$$(5) \quad \tilde{\Gamma}_{ij}^k = (g^T)^{k\bar{p}} \partial_i g_{p\bar{j}}^T.$$

Throughout this paper we will refer to these coordinates as *preferred local coordinates*.

**Definition 2.3.** A  $p$ -form  $\alpha$  on  $(S, \xi, \eta, \Phi, g)$  is called *basic* if  $\iota_\xi \alpha = 0$ , and  $\mathcal{L}_\xi \alpha = 0$ . In particular, a function  $f$  is said to be *basic* if  $\mathcal{L}_\xi f = 0$ . The ring of smooth basic functions will be denoted by  $C_B^\infty$ ; it is clearly a sub-ring of  $C^\infty$ .

The transverse complex structure  $\Phi$  allows us to decompose  $\Lambda_B^r \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda_B^{p,q}$ . We can then decompose  $d_B = \partial_B + \bar{\partial}_B$ , where  $\partial_B : \Lambda_B^{p,q} \rightarrow \Lambda_B^{p+1,q}$  and  $\bar{\partial}_B : \Lambda_B^{p,q} \rightarrow \Lambda_B^{p,q+1}$ . Any form  $\alpha \in \Lambda_B^{p,q}$  which satisfies  $\iota_\xi \alpha = 0$  can naturally be regarded as an element of  $\Lambda^{p,q}Q^*$ . The extra condition that  $\mathcal{L}_\xi \alpha = 0$  ensures that  $d\alpha \in \Lambda^{p+q+1}Q^*$ . In particular, we can regard  $\bar{\partial}_B$  on basic forms as a map  $\bar{\partial}_B : \Lambda^{p,q}Q^* \rightarrow \Lambda^{p,q+1}Q^*$ , and similarly for  $\partial$ . The transverse metric induces  $L^2$  inner products on each of these spaces and so, as in the Kähler case, we may define the  $\partial_B$  and  $\bar{\partial}_B$  Laplacians on the bundles  $\Lambda_B^{p,q}$ . For example, the  $\bar{\partial}_B$  Laplacian is given by  $\square_B = (g^T)^{kj} \nabla_k^T \nabla_j^T$ . We now say a brief word about volume forms and integration by parts on Sasaki manifolds. As noted before, the form  $(\frac{1}{2}d\eta)^n \wedge \eta$  is a non-vanishing  $(2n+1)$ -form, and hence defines a volume form on  $S$ . One can check that it agrees with the standard Riemannian volume form by looking in preferred local coordinates. Furthermore, in preferred local coordinates we have

$$d\mu := (\frac{1}{2}d\eta)^n \wedge \eta = (\sqrt{-1})^n n! \det(g_{k\bar{j}}^T) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \wedge dx.$$

Combining this with the local formula (5) for the Christoffel symbols of  $Q$ , we note that the standard integration by parts formulae from Kähler geometry hold for the bundle  $Q$ , its duals, tensor powers and wedge products. For example, we have;

**Proposition 2.1.** Let  $\phi \in \Lambda_B^{p,q-1}$ ,  $\psi \in \Lambda_B^{p,q}$  be two basic forms, then we have

$$\int_S (g^T)^{i\bar{j}} \nabla_j^T \phi_{\bar{B}A} \overline{\psi_{i\bar{D}C}} (g^T)^{\bar{C}A} (g^T)^{\bar{D}B} d\mu = - \int_S \phi_{\bar{B}A} \overline{(g^T)^{j\bar{i}} \nabla_j^T \psi_{i\bar{D}C}} (g^T)^{\bar{C}A} (g^T)^{\bar{D}B} d\mu$$

The proof of this proposition follows from the standard computation in the Kähler setting, and so we omit the details. The interested reader can find the computation in [21]. A transverse metric  $g^T$  is called a *transverse Einstein metric* if it satisfies  $Ric^T = \kappa g^T$  for some constant  $\kappa$ . In particular, we observe that it is necessary that the basic first Chern class be signed. The following proposition describes a well-known obstruction to the existence of a transverse Einstein metric. See, for example, [15].

**Proposition 2.2.** The basic first Chern class is represented by  $\kappa d\eta$  for some constant  $\kappa$  if and only if  $c_1(D) = 0$ , where  $D = \ker \eta$  is the contact subbundle.

If  $g^T$  is a transverse Einstein metric with Einstein constant  $\kappa$ , then one can check that the Sasaki metric  $g$  is  $\eta$ -Einstein, satisfying

$$Ric_g = (\kappa - 2)g + (2n + 2 - \kappa)\eta \otimes \eta.$$

The metric  $g$  is *Sasaki-Einstein* if  $Ric_g = \lambda g$ . In particular, a Sasaki manifold admits a Sasaki-Einstein metric if and only if it admits a transverse Einstein metric with Einstein constant equal  $2n+2$ . In fact, we shall see that if  $\kappa > -2$ , then any transverse Einstein metric can be deformed to a Sasaki-Einstein metric via the so called  $\mathcal{D}$ -homothetic transformations.

### 3. PERTURBATIONS OF SASAKI STRUCTURES AND THE SASAKI-RICCI FLOW

A Sasaki manifold has a large number of defining structures. It is natural to ask what happens when one perturbs these structures, individually or together. In this section we describe two types of deformations of Sasaki structures, the second of which motivates the Sasaki-Ricci flow. We begin by describing the  $\mathcal{D}$ -homothetic deformations introduced by Tanno [36]. For  $a > 0$ , the rescaling

$$g' = ag + (a^2 - a)\eta \otimes \eta, \quad \eta' = a\eta, \quad \xi' = a^{-1}\xi, \quad \Phi' = \Phi$$

gives a Sasaki structure  $(\xi', \eta', \Phi', g')$  with the same holomorphic structure on the cone, but with radial variable  $r' = r^a$ . One can check that if  $g^T$  is a transverse Einstein metric with Einstein constant  $c > 0$ , then the  $\mathcal{D}$ -homothetic deformation with  $a = c/2n$  gives a Sasaki-Einstein metric. Throughout this paper, we shall assume that  $c_1^B(S) > 0$ , and  $c_1(D) = 0$ . By making a  $\mathcal{D}$ -homothetic deformation we may always assume that  $(2n+2)[\frac{1}{2}d\eta]_B = c_1^B(S)$ . A second class of transformations is described in the following proposition, which is proved in [15, 34].

**Proposition 3.1.** *Fix a Sasaki manifold  $\mathcal{S} = (S, g, \eta, \xi, \Phi)$ . Then any other Sasaki structure on  $S$  with the same Reeb vector field  $\xi$ , the same complex structure on the cone  $C(S) = \mathbb{R} \times S$ , and the same transversely complex structure on the Reeb foliation is related to the original structure via the deformed contact form  $\eta' = \eta + 2d_B^c \phi$ , where  $\phi$  is a smooth basic function that is sufficiently small.*

Deformations of this type shall be referred to as deformations of type II, following [5]. This motivates the definition of the space of Sasaki potentials;

**Definition 3.1.** *Fix a Sasaki structure  $(\xi, \eta, \Phi)$ . We define the space of Sasaki potentials to be*

$$\mathcal{H}_\eta = \{\phi \in C_B^\infty : \tilde{\eta} = \eta + 2d_B^c \phi \text{ is a contact 1-form}\}.$$

*If  $\eta$  and  $\eta'$  are related by some  $\phi \in \mathcal{H}_\eta$ , then we will say that  $\eta'$  is in the Kähler class of  $\eta$ .*

Notice that under type II deformations the transverse metric is deformed by  $\tilde{g}^T = g^T + d_B d_B^c \phi$ . In [33] the flow

$$(6) \quad \frac{\partial g^T}{\partial t} = -\text{Ric}_{g^T(t)} + \kappa g^T(t),$$

was studied. We will refer to this flow as the *normalized* Sasaki Ricci flow, as the volume is fixed under the flow. When the initial metric satisfies  $\kappa g^T \in c_1^B(S)$ , it was shown that this flow can be reduced to a transversely parabolic complex Monge-Ampère equation on the Sasaki potential, given by

$$(7) \quad \frac{\partial}{\partial t} \phi = \log \det(g_{kl}^T + \partial_l \partial_{\bar{k}} \phi) - \log \det(g_{\bar{k}l}^T) + \kappa \phi - F,$$

where the function  $F$  is defined by  $\text{Ric}^T = \kappa d\eta_o + d_B d_B^c F$ , according the transverse  $\partial\bar{\partial}$ -lemma of [14]. It was proved in [33] that (6) is well-posed. By Proposition 3.1, the solution to equation (7) defines a one parameter family of Sasaki structures with the same transverse complex structure, the same Reeb field and the same complex structure on the cone. Moreover, it was shown that the solution  $\phi$  exists for all time, remains basic, and converges exponentially if  $c_B^1 \leq 0$ . The transverse  $\partial\bar{\partial}$ -lemma of [14] implies there is a basic function  $u : S \rightarrow \mathbb{R}$  so that

$$\partial_j \partial_{\bar{k}} \dot{\phi} = \dot{g}_{\bar{k}j}^T = -R_{\bar{k}j}^T + g_{\bar{k}j}^T = \partial_j \partial_{\bar{k}} u,$$

and so  $\phi$  evolves by  $\dot{\phi}(t) = u(t) + c(t)$ . We can use the function  $c(t)$  to adjust the initial value  $\phi(0)$ . As in the Kähler case, the specific choice for the initial value of  $\phi$  plays an important role in translating estimates for the transverse Kähler potential to estimates for the transverse metric  $g^T(t)$ , [25]. Below, we make the case that a similar choice for the initial value of  $\phi$  as in [25] is preferred in the Sasaki-Ricci flow. We first compute that

$$\dot{u} = \square_B u + u + a(t)$$

for some basic function  $a(t)$  depending only on  $t$ , which we fix by  $\int e^{-u} d\mu = \text{Vol}(S)$ . It will simplify matters to define a probability measure on  $S$  by  $d\rho := \text{Vol}(S)^{-1} e^{-u} d\mu$ . One can easily compute that

$$(8) \quad a(t) = \int_S u d\rho = \frac{1}{\text{Vol}(S)} \int_S u e^{-u} d\mu.$$

In [10] (see also [21] for another approach) transverse  $\mathcal{W}$  and  $\mu$  functionals were introduced, and shown to be monotone along the flow, thereby opening Perelman's methods to the Sasaki setting. The author applied these functionals to obtain a non-collapsing theorem for the Sasaki-Ricci flow, and to extend Perelman's uniform estimates for the Kähler-Ricci flow to the Sasaki setting. The precise results are;

**Proposition 3.2** ([10] Proposition 7.1). *Let  $g^T(s)$  be a solution of the normalized Sasaki-Ricci flow, and let  $\rho > 0$ . There exists a constant  $c > 0$ , depending only on  $g(0)$ , and  $\rho$  such that for every  $p \in S$ , and  $t \geq 0$*

$$\int_{\{y: d^T(p,y) < r\}} d\mu > cr^{2n}$$

**Theorem 3.1** ([10] Theorem 1.3). *Let  $g(t)$  be a solution of the normalized Sasaki-Ricci flow on a compact Sasaki manifold  $(S, \xi)$  of real dimension  $2n + 1$ , and transverse complex dimension  $n$ , with  $c_B^1(S) > 0$ . Let  $u \in C_B^\infty(S)$  be the transverse Ricci potential. Then there exists a uniform constant  $C$ , depending only on the initial metric  $g(0)$  so that*

$$(9) \quad |R^T(g(t))| + |u|_{C^1} + \text{diam}^T(S, g(t)) < C$$

where  $\text{diam}^T(S, g(t)) = \sup_{x,y \in S} d^T(x, y)$ .

It was pointed out in [21] that the transverse diameter bound in Theorem 3.1 implies a bound for the diameter. We include the argument for completeness.

**Lemma 3.1.** *There exists a constant  $C > 0$ , such that  $\text{diam}(S, g(t)) < C$  for all  $t$  along the Sasaki-Ricci flow.*

*Proof.* For every  $t \geq 0$ , we have  $g(t)(\xi, \xi) = 1$ . In particular, if  $p \in S$  has  $\text{orb}_\xi p$  closed, then the length of the curve defined by  $\text{orb}_\xi p$  is independent of  $t$ . Thus, the result is obvious in the regular and quasi-regular cases. When the Sasaki structure is irregular, the results of Rukimbira [30, 31] imply that there exists at least  $n + 1$  closed orbits of the Reeb field. Let  $p \in S$  be a point with closed orbit, and let  $\text{orb}_\xi p$  have length  $A$ . We have

$$d(x, y) \leq d^T(x, p) + A + d^T(y, p) \leq 2\text{diam}^T(S, g(t)) + A.$$

which completes the proof.  $\square$

We now discuss the choice for the initial value of the transverse Kähler potential. Suppose that a given flow  $\phi$  satisfies  $\phi(0) = c_0$ , then one can easily check that  $\tilde{\phi} := \phi + (\tilde{c}_0 - c_0)e^{\kappa t}$  satisfies the same flow with initial condition  $\tilde{c}_0$ . This underlines the importance of choosing the initial value properly; any two solutions with different initial value differ by terms diverging exponentially in time. We introduce the quantity

$$(10) \quad c_0 = \int_0^\infty e^{-t} \|\nabla \dot{\phi}\|_{L^2}^2 dt + \frac{1}{\text{Vol}(S)} \int_S u(0) d\mu_0.$$

One easily checks that this does not depend on  $\phi(0)$ . The first indication that this is the correct choice for  $\phi(0)$  is that the bound in Theorem 3.1 implies

$$(11) \quad \sup_{t \geq 0} \|\dot{\phi}\|_{C^0} \leq C.$$

To see this, observe that  $\partial_B \bar{\partial}_B(u - \dot{\phi}) = 0$ , and so by the uniform bound for  $u$  it suffices to bound,  $\alpha(t) := \text{Vol}(S)^{-1} \int_S \dot{\phi} d\mu_t$ . This is easily done by computing the evolution of  $\alpha$ ; see the computations in [25]. The upshot of this is contained in the following;

**Proposition 3.3.** *Let  $(S, \xi, \eta_0, \Phi, g_0)$  be a compact Sasaki manifold with  $\kappa[\frac{1}{2}d\eta_0]_B = c_1^B(S)$  for any constant  $\kappa$ . Consider the Sasaki-Ricci flow defined by (7). Then we have the a priori estimates*

$$\sup_{t \geq 0} \|\phi\|_{C^0} \leq A_0 < \infty \iff \sup_{t \geq 0} \|\phi\|_{C^k} \leq A_k < \infty \quad \forall k \in \mathbb{N}$$

The estimates in Proposition 3.3 are the transverse parabolic version of Yau's famous estimates [40], or equivalently the parabolic version of El Kacimi-Alaoui's generalization of Yau's estimates [14]. For the Kähler-Ricci flow, these estimates are well known, and can be found, for example, in [7, 25]. For the Sasaki-Ricci flow, the estimates can be found in [33]. In light of the uniform bound for  $\dot{\phi}$ , it is straight forward to check that Proposition 3.3 holds.

#### 4. BERNSTEIN-BANDO-HAMILTON-SHI ESTIMATES FOR THE TRANSVERSE CURVATURE

In this section we prove Theorem 1.3, which follows essentially from the arguments in the Riemannian case, and the curvature identities (2), (3), and (4). As the techniques used to prove this theorem in the Riemannian case are purely local, one hopes that this result will carry over to the Sasaki case. However, we must proceed with some care. For example, the standard commutation formulae for the covariant derivative and the Laplacian do not apply to 'tensors' on the bundle  $Q$ . In fact, the commutation relation which does hold involves not only the transverse Riemann tensor, but also the full Riemann curvature; here the curvature identities (2), (3), and (4) are crucial. In order to avoid being swamped by indices we introduce the following notation; if  $A$  and  $B$  are two sections of  $TS^{*\otimes p} \otimes Q^{*\otimes q}$ , we denote by  $A * B$  any quantity obtained from  $A \otimes B$  by summation over paired indices, contraction with  $g, g^{-1}, g^T$ , or  $(g^T)^{-1}$ , and multiplication by constants depending only on  $n, p$  and  $q$ .

**Lemma 4.1.** *Suppose  $A \in TS^{*\otimes p} \otimes Q^{*\otimes q}$ . Then there is a constant  $C$ , depending only on  $p, q$ , and  $n$ , such that the following commutation relation holds;*

$$[\nabla, \Delta]A = Rm^T * \nabla A + A * \nabla Rm^T + C\nabla A.$$

*Proof.* The argument is elementary, and so we only provide a sketch. The key point is that commuting covariant derivatives yields an expression involving both the full Riemann tensor, and the transverse Riemann tensor. In particular,

$$[\nabla, \Delta]A = Rm^T * \nabla A + Rm * \nabla A + A * \nabla Rm + A * \nabla Rm^T.$$

We now use the curvature relations (2), (3), and (4) to replace all the terms involving the full curvature tensor with terms involving only the metric and  $Rm^T$ . Collecting terms, and observing that any constants which appear depend only on  $n, p$  and  $q$ , the lemma is proved.  $\square$

*Proof of Theorem 1.3.* We begin by computing

$$(12) \quad \frac{\partial}{\partial t} Rm^T - \Delta Rm^T = Rm^T * Rm^T.$$

We can now compute the evolution equation for  $|\nabla Rm^T|^2$ . Before proceeding, we point out that the quantity  $|\nabla Rm^T|^2$  involves both  $g^T$  and  $g$ , as the covariant derivative  $\nabla$  takes arguments in  $TS$ . However, by looking in preferred coordinates we clearly have  $\nabla_{\frac{\partial}{\partial x}} Rm^T = 0$ , and so we see that the norm  $|\nabla Rm^T|^2$  agrees with the norm when we replace  $g$  by  $g^T$ , regarded as a bilinear form on  $TS$ . With this in mind, we obtain

$$(13) \quad \frac{\partial}{\partial t} |\nabla Rm^T|^2 = \Delta |\nabla Rm^T|^2 - 2|\nabla^2 Rm^T|^2 + Rm^T * (\nabla Rm^T)^{*2} + (\nabla Rm^T)^{*2}.$$

The last line follows by Lemma 4.1. We now consider the quantity  $F = t|\nabla Rm^T|^2 + \beta|Rm^T|^2$ . Using equations (12) and (13), we compute that

$$\begin{aligned} \frac{\partial}{\partial t} F &\leq |\nabla Rm^T|^2 + t \frac{\partial}{\partial t} |\nabla Rm^T|^2 + \beta(Rm^T)^{*3} + \beta \Delta |Rm^T| - 2\beta |\nabla Rm^T|^2 \\ &\leq \Delta F + (1 + c_1 t |Rm^T| + c_2 t - 2\beta) |\nabla Rm^T|^2 + c_3 |Rm^T|^3. \end{aligned}$$



By assumption,  $|Rm^T| \leq K$  if  $t \in [0, \alpha/K]$ . Set  $2\beta = 1 + c_1\alpha + c_2\frac{\alpha}{K}$ , then

$$\frac{\partial}{\partial t}F - \Delta F \leq c_3K^3.$$

Applying the maximum principle yields  $F(x, t) \leq \beta K^2 + c_3\beta K^3 t$ . Using the definition of  $\beta$  we obtain that there is a constant  $C_4$  depending only on  $n$ , and  $\max\{\alpha, 1\}$ , such that for every  $t \in (0, \alpha/K]$  we have

$$|\nabla Rm^T| \leq \sqrt{\frac{F}{t}} \leq C_4 t^{-1/2} \max\{K^{1/2}, K\}.$$

This proves the theorem in the case  $m = 1$ . The general case of  $m > 1$  follows by making similar adaptations to the Kähler, or Riemannian case. See [9] for details in these cases. The curvature equations (2), (3), and (4) show that our bounds for  $Rm^T$  extend to bounds for the full Riemann tensor.  $\square$

**Corollary 4.1.** *If  $Rm^T$  is uniformly bounded in  $C^0$  along the normalized Sasaki-Ricci flow, then for any time  $A > 0$ , there is a constant  $C_{A,k}$  depending only on  $|Rm^T|_{C^0}$ ,  $k$  and  $A$ , so that  $|Rm^T|_{C^k} \leq C_{A,k}$  for all  $t \geq A$ .*

## 5. THE $\bar{\partial}$ -OPERATOR ON FOLIATE VECTOR FIELDS AND STABILITY ON SASAKI MANIFOLDS

For the remainder of the paper we will be concerned with employing various types of stability to prove the convergence of the Sasaki-Ricci flow. We begin by presenting two notions of stability on Sasaki manifolds which generalize notions on Kähler manifolds. Later, we shall introduce the sheaf of transverse foliate vector fields, which seems central to the problem of stability on Sasaki manifolds. We first define the Futaki invariant of a Sasaki manifold. In the Sasaki case, the Lie algebra on which the Futaki invariant acts is the space of Hamiltonian, holomorphic vector fields.

**Definition 5.1** ([15] Definition 4.5). *Let  $U_\alpha = I \times V_\alpha$  be a foliated coordinate patch, with  $I \subset \mathbb{R}$  an open interval, and  $V_\alpha \subset \mathbb{C}^n$ . Let  $\pi_\alpha : U_\alpha \rightarrow V_\alpha$  be the projection. A complex vector field  $X$  on a Sasaki manifold is called a Hamiltonian holomorphic vector field if*

- (i)  $d\pi_\alpha$  is a holomorphic vector field on  $V_\alpha$
- (ii) the complex valued function  $u_X := \sqrt{-1}\eta(X)$  satisfies

$$\partial_{\bar{B}} u_X = -\frac{\sqrt{-1}}{2} \iota_X d\eta.$$

Such a function  $u_X$  is called a Hamiltonian function.

*Remark.* If  $X$  is a Hamiltonian holomorphic vector field, then in preferred local coordinates

$$(14) \quad X = \eta(X) \frac{\partial}{\partial x} + \sum_{i=1}^n X^i \frac{\partial}{\partial z^i} - \eta \left( \sum_{i=1}^n X^i \frac{\partial}{\partial z^i} \right) \frac{\partial}{\partial x},$$

where the  $X^i$  are local, holomorphic basic functions.

The Futaki invariant was originally defined for Sasaki manifolds by Boyer, Galicki and Simanca in [6], where it was considered to be a character on a quotient of the Lie algebra of “transversally holomorphic” vector fields (see [6] Definition 4.5). In [15] Futaki, Ono and Wang recast the Futaki invariant as a character of the Lie algebra on Hamiltonian holomorphic vector fields as defined above. For our purposes, we are only concerned with the case when the distribution  $D$  has  $c_1(D) = 0$ .

**Theorem 5.1** ([6] Proposition 5.1, [15] Theorem 4.9). *Let  $(S, g)$  be a Sasaki manifold with  $c_1^B(S) > 0$ , and  $c_1(D) = 0$ . Assume  $(2n+2)g^T \in c_1^B(S)$  and let  $u$  be the transverse Ricci potential for the metric  $g$ , and let  $X$  be a holomorphic, Hamiltonian vector field. We define the Futaki invariant  $fut_S(X)$  by the equation*

$$fut_S(X) := \int_S X u d\mu.$$

Then  $\text{futs}_S(X)$  is independent of the choice of Sasaki metric in  $(2n+2)^{-1}c_1^B(S)$ .

*Remark.* It is clear that the vanishing of the Futaki invariant is necessary for the existence of a Sasaki-Einstein metric.

Later in this section we will provide an alternative characterization of the Futaki invariant as a character on a certain subspace of the global sections of the soon-to-be-defined sheaf  $\mathcal{E}$ , and show that this characterization is equivalent to the above. We now introduce the Mabuchi energy on a Sasaki manifold in the special case that  $c_1(D) = 0$ ;

**Theorem 5.2** ([15] Theorem 4.12). *Let  $(S, g)$  be a Sasaki manifold, with  $c_1^B(S) = \kappa[d\eta_0]_B$ . Let  $\eta'$  be in the Kähler class of  $\eta_0$ . Let  $\phi_t$ ,  $t \in [a, b]$  be a path in  $\mathcal{H}_{\eta_0}$  connecting  $\eta, \eta'$ . Then the Mabuchi K-energy,*

$$\nu_{\eta_0}(\eta') = - \int_a^b \int_S \dot{\phi}_t (R^T - \bar{R}^T) d\mu_t dt,$$

*is independent of the path  $\phi_t$ .*

In the Kähler theory, the boundedness below of the Mabuchi K-energy is crucial to the existence of canonical metrics. It is known that a bound below for the K-energy is not sufficient to guarantee the existence of a Kähler-Einstein metric; a counterexample is given by Tian's unstable deformation of the Mukai-Umemura threefold [8, 12, 13, 38]. However, Bando and Mabuchi proved that any manifold that admits a Kähler-Einstein metric  $\omega$  necessarily has the K-energy bounded below on the Kähler class of  $\omega$  [2, 3]. Their result extends to Sasaki manifolds.

**Proposition 5.1.** *Suppose  $(S, \xi, \eta, \Phi, g)$  is a Sasaki manifold with  $c_1(D) = 0$ , and  $c_1^B(S) = (2n+2)[\frac{1}{2}d\eta]_B$ . Assume that  $S$  admits a Sasaki-Einstein metric in the Kähler class of  $g$ . Then the Mabuchi K-energy is bounded below on  $[\frac{1}{2}d\eta]_B$ .*

*Proof.* The proof is a consequence of the work of Nitta and Sekiya [24]. The estimates in [24] imply that the result of Bando and Mabuchi in [3] holds on Sasaki manifolds. That is, the Mabuchi K-energy is bounded below on the set of Sasaki metrics with positive transverse Ricci curvature. It is straight forward to check that the extension of the results of [3], due to Bando in [2] carries over verbatim to the Sasaki setting.  $\square$

If  $g^T(t)$  is evolving by the Sasaki-Ricci flow (6), then

$$(15) \quad \nu_{\eta_0}(\eta(a)) = \int_0^a \int_S u(R^T - \kappa n) d\mu_t dt = \int_0^a \int_S u(-\square_B u) d\mu_t dt = \int_0^a \int_S (g^T)^{j\bar{k}} \partial_{\bar{j}} u \overline{\partial_k u} d\mu_t dt.$$

The last term in the above expression is precisely the integral of the  $L^2$  norm of  $\partial_B u$  regarded as a section of  $\Lambda^{1,0}Q^*$ .

A significant difficulty in extending the results of [27] is identifying the appropriate operator to study and determining the domain on which to study it. As noted in the introduction, we are guided primarily by the model case of a regular Sasaki manifold.

**Definition 5.2.** *On an open subset  $U \subset S$ , let  $\Xi(U)$  be the Lie algebra of smooth vector fields on  $U$  and let  $\mathcal{N}_\xi(U)$  be the normalizer of the Reeb field in  $\Xi(U)$ ,*

$$\mathcal{N}_\xi(U) = \{X \in \Xi(U) : [X, \xi] \in L_\xi\}.$$

*We define a sheaf  $\mathcal{E}$  on  $S$  by*

$$\mathcal{E}(U) := \mathcal{N}_\xi(U)/L_\xi.$$

*The sheaf  $\mathcal{E}$  will be referred to as the sheaf of transverse foliate vector fields.*

When there is some chance of confusion, for  $V \in TS$  we denote by  $[V]$  the equivalence class of  $V$  in  $Q$ . Recalling the exact sequence (1), the inclusion  $\mathcal{N}_\xi \subset TS$  induces an inclusion of sheaves  $\mathcal{E} \subset Q$ . The sheaf  $\mathcal{E}$  is easily seen to be a locally free sheaf of  $C_B^\infty$ -modules. When the Reeb

field is regular or quasi-regular, the sheaf  $\mathcal{E}$  descends through the quotient to the sheaf of smooth sections of the tangent bundle of the Kähler manifold or orbifold.  $\mathcal{E}$  inherits a great deal of structure from the vector bundle  $Q$ ; the metric  $g^T$  restricts to a metric on  $\mathcal{E}$ , and it is easy to check that the transverse complex structure  $\Phi$  on  $Q$  restricts to an endomorphism of  $\mathcal{E}$ , and hence splits  $\mathcal{E}$  as  $\mathcal{E} = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ . The key observation is that  $\mathcal{E}$  has a well defined  $\bar{\partial}$  operator. Define a map  $\mathfrak{e} : Q \rightarrow Q$  by  $\mathfrak{e}(V) := \nabla_{\xi}^T V$ . By the definition of  $\nabla^T$ , it is clear that  $\mathcal{E}$  is precisely the kernel of the map  $\mathfrak{e}$ . In particular, on  $\mathcal{E}$  the covariant derivative  $\nabla^T$  descends to a well defined map  $d_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E} \otimes Q^*$ . This map is defined for  $[V] \in \mathcal{E}$  and  $[X] \in Q$  by

$$d_{\mathcal{E}}[V]([X]) = \nabla_X^T[V],$$

and this is clearly independent of the representative of the equivalence class  $[X]$ . Moreover, the transverse complex structure  $\Phi$  yields a splitting  $\mathcal{E} \otimes Q^* = \mathcal{E} \otimes (Q^*)^{1,0} \oplus \mathcal{E} \otimes (Q^*)^{0,1}$ . The map  $d_{\mathcal{E}}$  then splits as  $\partial_{\mathcal{E}} + \bar{\partial}_{\mathcal{E}}$ . Hence, we have a well defined operator  $\bar{\partial}_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E} \otimes (Q^*)^{(0,1)}$ . We can extend the operator  $\bar{\partial}_{\mathcal{E}}$  to a differential operator. Dualizing the exact sequence (1) we have an operator, also denoted by  $\bar{\partial}_{\mathcal{E}}$  satisfying

$$\bar{\partial}_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E} \otimes p^{\dagger}(Q^*) \hookrightarrow \mathcal{E} \otimes TS^*.$$

The metric  $g^T$  induces  $L^2$  inner products on  $\Gamma(X, \mathcal{E}^{1,0})$  and  $\Gamma(X, \mathcal{E}^{1,0} \otimes p^{\dagger}(Q^*))$ . We can then define the formal adjoint of the  $\bar{\partial}_{\mathcal{E}}$ -operator, denoted  $\bar{\partial}_{\mathcal{E}}^{\dagger}$ , on smooth sections by the usual formula. One easily computes using integration by parts that  $\bar{\partial}_{\mathcal{E}}^{\dagger}[W] = -(g^T)^{j\bar{k}} \nabla_j^T W_{\bar{k}}^p$ , and hence we can define the Laplacian of  $\mathcal{E}$  by

$$\square_{\mathcal{E}}[V] = -(g^T)^{j\bar{k}} \nabla_j^T \nabla_{\bar{k}}^T V^p.$$

We comment that at first glance this operator does not appear to be elliptic. However, by recalling the definition of the bundle  $\mathcal{E}$ , we see that

$$\square_{\mathcal{E}}[V] = -(\xi^* \otimes \xi^*) \nabla_{\xi}^T \nabla_{\xi}^T V^p - (g^T)^{j\bar{k}} \nabla_j^T \nabla_{\bar{k}}^T V^p,$$

which is clearly elliptic. Thus we can apply the usual elliptic theory. In a similar fashion we may also define the  $\partial_{\mathcal{E}}$  Laplacian  $\bar{\square}_E$ . A standard integration by parts computation proves;

**Proposition 5.2.** *For every  $[V] \in \Gamma(S, \mathcal{E}^{1,0})$ , we have the Bochner-Kodaira formula for the Laplacians  $\square_E$  and  $\bar{\square}_E$*

$$\|\partial_{\mathcal{E}}[V]\|_{L^2}^2 = \|\bar{\partial}_{\mathcal{E}}[V]\|_{L^2}^2 + \int_S R_{k\bar{j}}^T V^j \bar{V}^{\bar{k}} d\mu,$$

**Proposition 5.3.** *We define the space  $H^0(\mathcal{E}^{1,0})$ , which we refer to as the space global holomorphic sections of the sheaf of transverse foliate vector fields, by*

$$H^0(\mathcal{E}^{1,0}) := \text{Ker} \bar{\partial}_{\mathcal{E}}|_{\mathcal{E}^{(1,0)}}.$$

- $H^0(\mathcal{E}^{1,0})$  has the structure of a finite dimensional Lie algebra over  $\mathbb{C}$ .
- $H^0(\mathcal{E}^{1,0})$  is isomorphic as a Lie algebra to the Lie algebra of holomorphic, Hamiltonian vectorfields on  $S$ .
- The space  $H^0(\mathcal{E}^{1,0})$  depends only on the complex structure  $J$  on the cone, and the Reeb field  $\xi$ , and the transverse holomorphic structure. In particular,  $\dim H^0(\mathcal{E}^{1,0})$  is invariant along the Sasaki Ricci flow.

*Remark.* The Lie algebra  $H^0(\mathcal{E}^{1,0})$  has appeared in the literature before, under a number of different guises. In [23] it was proved that if the transverse scalar curvature of the Sasaki metric  $g$  is constant, then  $H^0(\mathcal{E}^{1,0})$  is reductive.  $H^0(\mathcal{E}^{1,0})$  also played an important role in the work of Boyer, Galicki and Simanca on extremal Sasaki metrics [6]. In [24] it was shown that if  $S$  admits a Sasaki-Einstein metric  $g_{SE}$ ,  $\mathcal{G}$  is the identity component of the automorphism group of the Sasaki structure, and  $\mathcal{O}$  is the orbit of  $g_{SE}$  under the action of  $\mathcal{G}$ , then the tangent space to  $\mathcal{O}$  at the point  $g_{SE}$  is isomorphic to  $H^0(\mathcal{E}^{1,0})$ .

We delay the proof of Proposition 5.3 for a moment in order that we may discuss the local structure of  $\mathcal{E}$ . We feel this local picture is essential to understanding the structure we are describing abstractly and so we shall be very explicit in this part of the development. Let  $U \subset S$  be an open subset of  $S$  on which we have a preferred coordinate system, and suppose that  $[V] \in \Gamma(U, \mathcal{E})$  is a section of  $\mathcal{E}$  over  $U$ . Over  $U$  we can write  $V = V^i[\partial_{z^i}] + V^{\bar{i}}[\partial_{\bar{z}^i}]$ . Let  $V$  be any lift of  $[V]$  to  $\mathcal{N}_\xi(U)$ . In preferred local coordinates

$$V = f \frac{\partial}{\partial x} + \sum_{i=1}^n V^i \frac{\partial}{\partial z^i} + \sum_{i=1}^n V^{\bar{i}} \frac{\partial}{\partial \bar{z}^i}.$$

As  $[V] \in \mathcal{E}$ , we necessarily have  $V^i, V^{\bar{i}}$  basic functions. If we restrict to  $[V] \in \mathcal{E}^{1,0}$ , then  $V^{\bar{i}} = 0$ . A basis of  $Q^{0,1}$  over  $U$  is given by  $\{[\partial_{\bar{z}^j}]\}$  where  $1 \leq j \leq n$ . By definition, we compute

$$(16) \quad \bar{\partial}_\mathcal{E}[V] = \sum_{i=1}^n \partial_{\bar{z}^j} V^i \left[ \frac{\partial}{\partial z^i} \right] \otimes d\bar{z}^j.$$

*Proof of Proposition 5.3.* Since  $S$  is compact, the kernel of the self-adjoint elliptic operator  $\square_\mathcal{E}$  is finite dimensional, and hence  $H^0(\mathcal{E}^{1,0})$  is a finite dimensional vector space over  $\mathbb{C}$ . The Jacobi identity shows that the Lie bracket on  $\Xi(S)$  descends to a Lie bracket on  $\Gamma(U, \mathcal{E})$ . Moreover, the local formulae show that the bracket preserves the decomposition  $\mathcal{E} = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ . Computing over an open set shows that the Lie bracket preserves the space  $H^0(\mathcal{E}^{1,0})$ , and establishes the first statement. We turn our attention to the second and third statements. Suppose  $[V] \in H^0(\mathcal{E}^{1,0})$ , then equation (16) shows that condition (i) of Definition 5.1 is satisfied for any lift  $V_c := \sigma(V) + c\xi$  of  $[V]$ . It remains to show condition (ii) holds for some choice of  $c$ . Define  $\tilde{c} := -\xi^*(\sigma([V]))$ . Using the formula for  $\eta$  in preferred local coordinates yields  $u_{V_c} = -\sum_{i=1}^n h_i V^i$ . As  $\bar{\partial}_\mathcal{E}[V] = 0$ , equation (16) shows that  $V_c$  satisfies property (ii) of Definition 5.1. Conversely, suppose  $V$  is a Hamiltonian, holomorphic vector field. The expression (14) for  $V$ , along with the remarks following Definition 5.1 show that  $V$  is in the normalizer of  $\xi$  and that  $[V] \in H^0(\mathcal{E}^{1,0})$ . Computing locally we see that the Lie bracket commutes with these identifications. Finally, by Proposition 3.1 any Sasaki structure on  $S$  with the same complex structure  $J$  on the cone, Reeb field  $\xi$  and transverse holomorphic structure is related to the original Sasaki structure by deformations of type II. One can then easily check via the above computations that  $H^0(\mathcal{E}^{1,0})$  is unchanged under these deformations.  $\square$

**Corollary 5.1.** *The Futaki invariant defines a character on the Lie algebra  $H^0(E^{1,0})$ .*

*Remark.* Corollary 5.1 is essentially a restatement of the original definition in [6]. However, the identification of the Lie algebra which appears in [6] as the global holomorphic sections of the sheaf  $\mathcal{E}$  provides what we feel to be a particularly attractive definition of the Futaki invariant, which is seen to generalize the Kähler setting.

**Proposition 5.4.** *Let  $(S, g_0)$  be a Sasaki manifold with  $c_1^B(S) > 0$ , and  $c_1(D) = 0$ . Suppose that  $(2n+2)[\frac{1}{2}d\eta_0]_B = c_1^B(S)$ . If  $\nu_{\eta_0}(\eta) > -\infty$  for every  $\eta$  in the Kähler class of  $\eta_0$ , then  $fut_S(X) \equiv 0$ .*

*Proof.* Suppose  $\nu_{\eta_0}(\eta) > -C > -\infty$ , but  $fut([X]) \neq 0$  for some  $[X] \in H^0(\mathcal{E}^{1,0})$ . We can assume that  $Re(fut([X])) < 0$ , by replacing  $[X]$  with  $[-X]$ , or  $[iX]$ . Let  $X = \sigma_0([X])$ . The vector field  $X$  is foliate, orthogonal  $\xi$ , and holomorphic. In preferred local coordinates we have  $X = \sum_{i=1}^n X^i(\frac{\partial}{\partial z^i} - \sqrt{-1}h_j \frac{\partial}{\partial x})$  for  $X^i$  basic, holomorphic functions, and  $h$  a local, real valued function. Define the vector field  $\tilde{X}$  on the cone  $C(S)$ , by  $\tilde{X}(z, r) = Re(X)(z)$ , where  $z \in S$ , and  $r$  is the radial variable on the cone. The local formula shows  $\tilde{X}$  is real holomorphic, and so  $\mathcal{L}_{\tilde{X}}J = 0$ , where  $J$  is the complex structure on the cone. Let  $\rho_t$  be the local flow of  $Re(X)$  on  $S$ , and  $\tilde{\rho}_t$  be the local flow of  $\tilde{X}$ . Then  $\tilde{\rho}_t$  is a biholomorphism, and it is clear that  $\tilde{\rho}_t(z, r) = (\Phi_t(z), r)$ . In particular,  $(S, \rho_t^*g)$  is a Sasaki manifold with the same Reeb field, the same complex structure on the cone and the same transversely holomorphic structure on the Reeb foliation. By Proposition 3.1 we have that  $\rho_t^*d\eta = d\eta + \partial_B \partial_{\bar{B}} \psi_t$ . We can now follow the argument in [37] to obtain the proposition.  $\square$

Our primary concern will be sections of  $\mathcal{E}$  induced from basic functions. In order to have our theory sufficiently well adapted for our future applications we discuss this now. Given a basic function  $h$ ,  $\bar{\partial}_B h$  is a section of  $\Lambda_B^{0,1}$ . We then define  $V^j = (g^T)^{j\bar{k}} \partial_{\bar{k}} h$ .  $V$  defines a section of the quotient bundle  $Q$ , and the splitting map  $\sigma$  satisfies  $\sigma([V]) = V$ . Moreover,  $V$  lies in the normalizer of  $\xi$  in  $TS$ . Thus,  $[V]$  defines a global section of  $\mathcal{E}^{1,0}$  over  $S$ . We now compute that

$$\partial_{\mathcal{E}}[V] = \sum_{l=1}^n \nabla_l^T \left( (g^T)^{j\bar{k}} \partial_{\bar{k}} h \right) \left[ \frac{\partial}{\partial z^j} \right] \otimes dz^l = \sum_{l=1}^n (g^T)^{j\bar{k}} \partial_l \partial_{\bar{k}} h \left[ \frac{\partial}{\partial z^j} \right] \otimes dz^l,$$

and so

$$(17) \quad \|\partial_{\mathcal{E}}[V]\|_{L^2(\mathcal{E}^{1,0} \otimes TS^*)} = \int_S (g^T)^{l\bar{p}} (g^T)^{j\bar{k}} (\nabla_l^T \nabla_{\bar{k}}^T h) \left( \overline{\nabla_p^T \nabla_j^T h} \right) d\mu.$$

Expressions such as these shall appear repeatedly in what is to follow. In order to simplify our notation, we will use  $\nabla$  and  $\bar{\nabla}$  to denote covariant derivative in the unbarred and barred directions. For example, equation (17) can then be written as  $\|\partial_{\mathcal{E}}[V]\|_{L^2} = \int_S |\nabla \nabla h|^2 d\mu$ .

## 6. PROOF OF THEOREM 1.1 PART (i)

In this section we use the bound below for the Mabuchi functional to show that the  $L^2$  norm of  $\partial_B u$  goes to zero as  $t \rightarrow \infty$ , where  $u$  is the transverse Ricci potential,  $R_{\bar{k}j}^T - g_{\bar{k}j}^T = \partial_j \partial_{\bar{k}} u$ . The uniform bounds for the transverse Riemann tensor allow us to employ an inductive argument to obtain the decay to zero of all Sobolev norms of  $\partial_B u$ . The Mabuchi K-energy along the normalized Sasaki-Ricci flow is given by (15). Thus, if the Mabuchi energy is bounded below on  $\mathcal{H}_{\eta_0}$  then there exists times  $t_k \rightarrow \infty$  such that  $\|\nabla u\|_{L^2(t_k)} \rightarrow 0$ . We can obtain convergence for the full sequence by computing the evolution equation for the quantity  $Y(t) = \|u\|_{L^2}$ . Following the computations in [27] we obtain

$$(18) \quad \dot{Y}(t) = (n+1)Y(t) - \int_S |\partial_B u|^2 R^T d\mu - \int_S |\bar{\nabla} \nabla u|^2 d\mu - \int_S |\nabla \nabla u|^2 d\mu.$$

Applying the uniform bound for  $R^T$  in Theorem 3.1, the argument from [27] carries over verbatim to yield;

**Lemma 6.1.** *Assume the Mabuchi K-energy is bounded from below on the Kähler class of  $\eta_0$ . Then  $Y(t) \rightarrow 0$  along the Kähler-Ricci flow as  $t \rightarrow \infty$ .*

*Proof of Theorem 1.1 part (i).* In light of Lemma 6.1, rearranging (18) and integrating with respect to  $t$  gives,

$$\int_0^\infty dt \int_S |\bar{\nabla} \nabla u|^2 d\mu_t + \int_0^\infty \int_S |\nabla \nabla u|^2 d\mu_t < \infty.$$

We are in a position to apply our previous argument inductively. Define  $Y_{r,s}(t) = \int_S |\nabla^s \bar{\nabla}^r u|^2 d\mu_t$ . Following the computations in [27], and making use of the uniform  $C^\infty$  bounds on  $Rm^T$  and  $Rm$  guaranteed by Theorem 1.3, we compute that

$$(19) \quad \begin{aligned} \dot{Y}_{r,s}(t) \leq & C_1 Y_{r,s}(t) + C_2 \left( \int_S |D^{r+s-p} u|^2 d\mu_t \right)^{1/2} Y_{r,s}^{1/2}(t) \\ & - \int_S |\nabla^{s+1} \bar{\nabla}^r u|^2 d\mu_t - \int_S |\bar{\nabla} \nabla^s \bar{\nabla}^r u|^2 d\mu_t, \end{aligned}$$

where summation over  $1 \leq p \leq r+s-1$  is understood. We now employ the argument in [27], which carries over verbatim.  $\square$

## 7. CONVERGENCE IN PRESENCE OF STABILITY

We begin this section by manipulating the equation (18) into a more suggestive form.

$$(20) \quad \dot{Y}(t) = - \int_S |\partial_B u|^2 (R^T - n) d\mu_t - \int_S \nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j}^T - g_{\bar{k}j}^T) d\mu_t - 2 \int_S |\bar{\nabla} \nabla u|^2 d\mu_t.$$

This follows by applying the Bochner-Kodaira formula obtained in Proposition 5.2 to the section of  $\mathcal{E}$  defined by  $V^j = (g^T)^{j\bar{k}} \partial_{\bar{k}} u$ . For large time  $t$ , the first two terms on the right hand side can easily be bounded by  $\epsilon Y$ , by Theorem 1.1 part (i). In order to obtain the exponential decay of the quantity  $Y$ , we must bound the last term in equation (20). Let  $\lambda_t$  be the smallest, strictly positive eigenvalue of the Laplacian  $\square_{\mathcal{E},t}$ . We include the subscript  $t$  to enforce that the metric  $g(t)$  is evolving. By the elliptic theory we have

$$\lambda_t \|V - \pi_t V\|_{L^2(\mathcal{E}^{1,0})} \leq \int_S |\bar{\partial}_{\mathcal{E}} V|^2 d\mu_t,$$

where  $\pi_t$  is the  $L^2$  projection onto  $H^0(\mathcal{E}^{1,0})$  with respect to the metric  $g^T(t)$ . As in the Kähler case, we observe that for the section  $V \in \mathcal{E}$  in question, we have by Corollary 5.1

$$\|\pi_t V\|^2 = f u t_s (\pi_t V).$$

Thus, equation (20) yields the inequality

$$(21) \quad \begin{aligned} \dot{Y} \leq & -2\lambda_t Y + 2f u t_s \left( \pi_t (g^T)^{j\bar{k}} \partial_{\bar{k}} u \right) \\ & - \int_S |\partial_B u|^2 (R^T - n) d\mu - \int_S \nabla^j u \nabla^{\bar{k}} u \left( R_{\bar{k}j}^T - g_{\bar{k}j}^T \right) d\mu \end{aligned}$$

We remark that equation (21) is completely general, and we view it as the motivating inequality for the developments in this paper.

*Proof of Theorem 1.1 part (ii).* Since the Mabuchi functional is bounded below, Proposition 5.4 implies that the Futaki invariant is zero. The uniform transverse curvature bound, and the conclusion of Theorem 1.1 part (i) imply that for any  $\epsilon > 0$ , there is a  $T_\epsilon$  such that, for every  $t \in [T_\epsilon, \infty)$  we have  $\dot{Y}(t) \leq (-\lambda_t + \epsilon)Y(t)$ . It suffices to find a positive lower bound for  $\lambda_t$ . Condition (C) is tailor made for the task.

**Theorem 7.1.** *Let  $(S, \xi, \eta, \Phi, g)$  be a compact Sasaki manifold of dimension  $2n + 1$ . Assume that the Sasaki structure  $(\xi, \eta, \Phi, g)$  satisfies stability condition (C). Fix  $V, D, \delta > 0$ , and constants  $C_k$ . Then there exists an integer  $N$  and a constant  $C(V, D, \delta, C_k, n, N) > 0$  such that*

$$C \|V\|^2 \leq \|\bar{\partial}_{\mathcal{E}} V\|^2, \quad \forall \quad V \perp H^0(\mathcal{E}^{1,0})$$

*for all Sasaki structures  $(S, \xi, \eta, \Phi, g)$  with  $\text{Vol}_g(S) < V$ , and  $\text{diam}_g(S) < D$ , and whose injectivity radius is bounded below by  $\delta$ , and the  $k$ -th derivative of whose curvature tensors are uniformly bounded by  $C_k$  for all  $k \leq N$ .*

The proof of Theorem 7.1 is taken up in the next section. It follows that for  $t$  sufficiently large, we have  $Y(t) \leq C e^{-ct}$ . With the exponential decay of the  $L^2$  norm of  $|\nabla u|$  established, a straight forward adaptation of the arguments in [27] yield the exponential decay of the  $L^2$  norms of  $\bar{\nabla}^r \nabla^s u$  where all norms are computed with respect to the evolving metric  $g^T(t)$ . The Sobolev imbedding theorem with uniform constants then gives the exponential decay of the  $C^k$  norm of  $u$  for any  $k$ . Since  $\dot{g}_{\bar{k}j}^T = \partial_j \partial_{\bar{k}} u$ , we have  $\int_0^\infty \sup_S |\dot{g}_{\bar{k}j}^T|_{g^T(t)} dt < \infty$ . A lemma of Hamilton ([20] Lemma 14.2) allows us to conclude that the metrics  $g_{\bar{k}j}^T$  on  $Q$  are uniformly equivalent. While Hamilton's proof is for metric tensors on the tangent bundle, one can easily check that the argument holds for vector bundles. The uniform equivalence of the metrics  $g^T$  imply that for any section  $W \in Q$  we have

$$|g_{\bar{k}j}^T(\mathcal{T}) W^j \overline{W^k} - g_{\bar{k}j}^T(\mathcal{S}) W^j \overline{W^k}| \leq C |W|_{t=0}^2 (e^{-c\mathcal{T}} - e^{-c\mathcal{S}})$$

As the last term goes to zero exponentially as  $\mathcal{S}, \mathcal{T} \rightarrow \infty$ , we obtain the exponential convergence of the transverse metrics to some metric  $g_{k\bar{j}}^T$  which is equivalent to all the metrics  $g_{k\bar{j}}^T(t)$ . Iteration yields exponential convergence in  $C^\infty$ . Since  $\partial_j \partial_{\bar{k}} h$  tends to zero, the limiting metric  $g_{k\bar{j}}^T(\infty)$  is Sasaki-Einstein.  $\square$

## 8. COMPACTNESS THEOREMS AND THE PROOF OF THEOREM 7.1

Our main objective in this section is to prove Theorem 7.1, which will finish the proof of Theorem 1.1. We begin by stating and proving a Sasaki version of Gromov compactness. This theorem is well known, and follows easily from Hamilton's compactness theorem [18] but we include the short proof for completeness.

**Theorem 8.1.** *Let  $(S, g)$  be a compact Sasaki manifold. Let  $g(t)$  be a sequence of Sasaki metrics on  $S$ , and  $J(t)$  a sequence of complex structures on the cone  $C(S)$  such that  $(C(S), \bar{g}(t) := dr^2 + r^2 g(t), J(t))$  is Kähler. Assume that the  $g(t)$ 's have bounded geometry in the sense that their volumes, diameters, curvatures and covariant derivatives of their curvature tensor are all bounded from above, and their injectivity radii are bounded below. Then there exists a subsequence  $\{t_j\}$ , and a sequence of diffeomorphisms  $F_{t_j} : S \rightarrow S$  such that the pulled back metrics  $F_{t_j}^* g(t_j)$  converge in  $C^\infty$  to a smooth metric  $\tilde{g}(\infty)$ . Moreover, on the cone, the lifted diffeomorphisms defined by  $\tilde{F}_{t_j}(r, z) := (r, F_{t_j}(z))$  have that the sequence  $\tilde{F}_{t_j}^* J(t_j)$  converges in  $C^\infty$  to an integrable complex structure  $\tilde{J}(\infty)$  on  $C(S)$ . Furthermore, the metric  $\tilde{g}(\infty)$  is Sasaki with respect to the complex structure  $\tilde{J}(\infty)$ . In particular, the Sasaki structures  $(\xi(t_j), \eta(t_j), \Phi(t_j), g(t_j))$  converge in  $C^\infty$  to a Sasaki structure  $(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ .*

*Proof.* The  $C^\infty$  convergence part of this theorem is just Hamilton's compactness theorem [18]. Thus, we are reduced to showing that the complex structures converge. This follows essentially from the proof of [27] Theorem 4, with the wrinkle that the cone  $C(S)$  is not compact. Consider instead the truncated cone  $\tilde{C}(S) = ((\frac{1}{2}, 1) \times S)$ . The argument in [27] shows that the complex structures converge to an integrable complex structure  $\tilde{J}(\infty)$  on compact sets making  $(\tilde{C}(S), \tilde{J}(\infty), dr^2 + r^2 \tilde{g}(\infty))$  into a Kähler manifold. We extend the complex structure to the whole cone by using the fact that  $\mathcal{L}_{r\partial_r} J(t_j) = 0$ , and that  $\tilde{F}_{t_j}^* r \partial_r = r \partial_r$ .  $\square$

*Proof of Theorem 7.1.* Let  $\lambda_t$  be the smallest positive eigenvalue of  $\square_{\mathcal{E}}$ , defined with respect to the Sasaki structure  $\mathfrak{s}(t) := (\xi(t), \eta(t), \Phi(t), g(t))$ . Suppose  $\mathfrak{s}(t)$  converges in  $C^\infty$  to a Sasaki structure  $\mathfrak{s}_\infty := (\xi_\infty, \eta_\infty, \Phi_\infty, g_\infty)$  and the dimension of the space of global holomorphic sections of the sheaf of transverse foliate vector fields is the same for every  $N \leq t \leq \infty$ . The perturbation theory for the Laplacian used in the proof of Theorem 3 extends to the global sections of the sheaf  $\mathcal{E}$ , and we obtain

$$(22) \quad \lim_{t \rightarrow \infty} \lambda_t = \lambda_\infty.$$

We can now prove by contradiction: assume there exists a sequence of metrics  $g(t)$  with  $\lambda_t \rightarrow 0$ . By passing to a subsequence (not relabeled), we can apply Theorem 8.1 to obtain the existence of diffeomorphisms  $F_t$  so that the pulled back Sasaki structure

$$\tilde{\mathfrak{s}}(t) := (F_t^* g(t), F_t^* \xi(t), F_t^* \eta(t), F_t^* \Phi(t))$$

converges in  $C^\infty$  to a Sasaki structure  $\tilde{\mathfrak{s}}_\infty = (\tilde{\xi}_\infty, \tilde{\eta}_\infty, \tilde{\Phi}_\infty, \tilde{g}_\infty)$ . By equation (22), the lowest positive eigenvalue of  $\tilde{\mathfrak{s}}(t)$  converges to a strictly positive limit. Let  $\mathcal{E}(t), \tilde{\mathcal{E}}(t)$  be the sheaves defined by the Sasaki structures  $\mathfrak{s}(t), \tilde{\mathfrak{s}}(t)$  respectively. Observe that the diffeomorphism  $F_t$  induces an isomorphism of sheaves  $\mathcal{E}(t) \cong \tilde{\mathcal{E}}(t)$ . Moreover,  $F_t$  is an isometry which preserves the transverse holomorphic structure, and hence descends to an isometry of the quotient bundles  $Q(t), \tilde{Q}(t)$ . Using the computations in §5, it is then clear that the sheaves  $\mathcal{E}(t), \tilde{\mathcal{E}}(t)$  are isospectral, providing a contradiction.  $\square$

## 9. THE PROOF OF THEOREM 1.2

Note that in the proof of Theorem 1.1 we only needed a bound on the smallest positive eigenvalue of  $\square_{\mathcal{E}}$  restricted to sections of  $\mathcal{E}^{1,0}$  induced by basic functions. Rather than study the  $\bar{\partial}_{\mathcal{E}}$  Laplacian on global sections of  $\mathcal{E}^{1,0}$  we are thus motivated to study the following operator on  $C_B^{\infty}$ ;

$$L := -(g^T)^{j\bar{k}} \nabla_j \nabla_{\bar{k}} + (g^T)^{j\bar{k}} \nabla_j u \nabla_{\bar{k}}.$$

Our developments will require the basic Sobolev and Lebesgue spaces, for which we refer the reader to [11]. The operator  $L$  appeared in [10], where it was shown to be elliptic and self-adjoint with respect to the  $L^2$  inner-product on  $H_B^2$  induced by the probability measure  $d\rho$  defined in §3. Moreover, it was shown that  $L$  has a complete spectrum of smooth, basic eigenfunctions  $\{\psi_j\}_{j \in \mathbb{N}}$  spanning  $L_B^2$ , with eigenvalues  $\lambda_j \geq 1$ . This yields the Poincaré inequality;

**Lemma 9.1.** *Let  $u$  satisfy the equation  $g_{\bar{k}j}^T - Ric_{\bar{k}j}^T = \partial_j \partial_{\bar{k}} u$ . Then the following inequality*

$$\frac{1}{Vol(S)} \int_S f^2 e^{-u} d\mu \leq \frac{1}{Vol(S)} \int_S |\nabla f|^2 e^{-u} d\mu + \left( \frac{1}{Vol(S)} \int_S f e^{-u} d\mu \right)^2$$

*holds for all  $f \in C_B^{\infty}(S)$ .*

Observe that if  $\lambda_t \geq 1 + \delta > 1$ , then a standard computation for the operator  $L$  suggests that the final term in (20) can be controlled by  $-\delta Y(t)$  (cf. equation (29) in [10]). However, as we are not assuming a lower bound for the Mabuchi energy, it is no longer natural to work with  $Y(t)$ . Instead, we define

$$W(t) := \frac{1}{Vol(S)} \int_S (u - a)^2 e^{-u} d\mu, \quad Z(t) := \frac{\partial}{\partial t} a = \frac{1}{Vol(S)} \int_S (|\nabla u|^2 - (u - a)^2) e^{-u} d\mu,$$

where  $a(t)$  is defined by (8). The main result in this section is the following proposition.

**Proposition 9.1.** *Assume that condition (F) holds on  $(n+1)^{-1}c_1^B(X)$ , and that condition (T) holds along the Sasaki-Ricci flow with initial value  $g_0$ . Then there are constants  $b, C > 0$  independent of  $t$  so that  $W(t) \leq C e^{-bt}$ , for every  $t \in [0, \infty)$ . Moreover,*

$$\|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R^T - (2n+2)n\|_{C^0} \leq C e^{-\frac{1}{2(n+1)}bt}, \quad t \in [0, \infty).$$

The proof of Proposition 9.1 proceeds in several steps. First, we describe a smoothing lemma which will reduce the proof of the proposition to proving the exponential decay of  $W$ . The important idea of a smoothing lemma was first introduced by Bando [2], and appeared in [28]; it was subsequently improved in [22].

**Lemma 9.2.** *There exist positive constants  $\delta, K$  depending only on  $n$  with the following property; for any  $\epsilon \in (0, \delta]$ , and any  $t_0 > 0$ , if  $\|u(t_0)\|_{C^0} \leq \epsilon$ , then*

$$\|\nabla u(t_0 + 2)\|_{C^0} + \|R(t_0 + 2) - (2n+2)n\|_{C^0} \leq K\epsilon.$$

The proof of Lemma 9.2 is identical to the Kähler case, and can be found in [28]. In order to prove Proposition 9.1, we must first establish the exponential decay of  $W$ . We now describe a condition under which such decay holds.

**Proposition 9.2.** *Suppose there is a uniform constant  $\delta > 0$  independent of time so that*

$$(23) \quad \frac{(1+\delta)}{V} \int_S (u(t) - a(t))^2 e^{-u(t)} d\mu_t \leq \frac{1}{V} \int_S |\nabla u(t)|^2 e^{-u(t)} d\mu_t.$$

*Then there are constants  $b, C > 0$  independent of  $t$  so that  $W(t) \leq C e^{-bt}$ .*

The proof of this proposition requires the following lemma, which is a consequence of the developments in [10].



**Lemma 9.3.** *The transverse Ricci potential  $u(t)$ , and its average  $a(t)$  satisfy the following inequalities, where the constants  $C_1$  and  $C_2$  depend only on  $g_0$ .*

- (i)  $0 \leq -a \leq \|u - a\|_{C^0}$
- (ii)  $\|u - a\|_{C^0}^{n+1} \leq C_1 \|\nabla u\|_{C^0}^2 \|u - a\|_{L^2} \leq C_2 \|\nabla u\|_{L^2} \|\nabla u\|_{C^0}^n$

*Proof.* We combine Lemma 9.1, Proposition 3.2 and follow the proof in [28].  $\square$

*Proof of Proposition 9.3.* The proof follows essentially from the arguments in [42], and so we provide only a sketch. We first claim that  $W(t) \rightarrow 0$  as  $t \rightarrow \infty$ . To see this, observe that  $Z(t) \geq \delta W(t)$  by assumption. From Theorem 3.1 we have

$$\int_0^\infty Z(t) dt = \lim_{t \rightarrow \infty} a(t) - a(0) < \infty.$$

Thus,  $Z(t) \rightarrow 0$  along a subsequence  $t_k \rightarrow \infty$ . The convergence of the full sequence is obtained as in §6, by computing the evolution equation for  $W$ . Lemma 9.3, combined with the uniform bounds in Theorem 3.1 imply that  $\|u - a\|_{C^0} \leq AW(t)^{1/(2n+2)}$ , and so  $u \rightarrow a$  in  $C^0$  as  $t$  goes to infinity. The result follows from elementary modifications to the proof of Lemma 2.4 in [42].  $\square$

Lemma 9.3 and the uniform bounds for  $u$  in Theorem 3.1 imply that if  $W(t)$  decays exponentially, then  $\|u\|_{C^0}$  decays exponentially. By Lemma 9.2, we see that the second statement in Proposition 9.1 follows from the exponential decay of  $W(t)$ . Our task is now reduced to showing that when conditions (T) and (F) hold, the assumptions of Proposition 9.3 are satisfied. We begin by showing that if the Futaki invariant vanishes and we have a non-degeneracy condition on the ‘second’ eigenvalue of  $L$ , then (23) holds.

**Proposition 9.3.** *Let  $\nu(t)$  be the smallest eigenvalue of  $L$  larger than one. Assume that  $\nu(t) \geq 1 + \delta$  for some  $\delta > 0$  uniformly along the flow. If the Futaki invariant vanishes, then (23) holds.*

*Proof.* Fix a time  $t$ , and from now on suppress the  $t$  variable. Recall that in §5 it was pointed out the  $(g^T)^{i\bar{j}} \partial_{\bar{j}} u$  defines a section of  $\mathcal{E}^{1,0}$ . For simplicity we denote this section by  $\nabla u \in \mathcal{E}^{1,0}$ . Since  $fut_S \equiv 0$ , we necessarily have  $\nabla u \perp H^0(\mathcal{E}^{1,0})$  in  $L^2(\mathcal{E}^{1,0}, d\mu)$ . Let  $\pi$  denote the orthogonal projection to  $H^0(\mathcal{E}^{1,0})$  in the space in  $L^2(\mathcal{E}^{1,0}, d\rho)$ . We decompose  $\nabla u = \pi(\nabla u) + V$ , then we have

$$(24) \quad \langle \pi(\nabla u), \pi(\nabla u) \rangle_{L^2(\mathcal{E}^{1,0}, d\mu)} \leq \langle V, V \rangle_{L^2(\mathcal{E}^{1,0}, d\mu)}.$$

Let  $\nu_0 = 1 < \nu_1 < \nu_2 < \dots$  be the distinct eigenvalues of  $L$  acting on the function space  $H_B^2(d\rho)$ , and let  $E_k$  be the eigenspace of  $\nu_k$ .  $E_0$  and may be empty, and corresponds to those basic functions which induce sections of  $H^0(\mathcal{E}^{1,0})$ . Let  $u - a = u_0 + u_1 + \dots$  be the unique decomposition of  $u - a$  into eigenfunctions  $u_k \in E_k$  so that  $u_i$  and  $u_j$  are orthogonal in  $L^2(d\rho)$  for  $i \neq j$ . For  $k \geq 1$  we have  $\nu_k \geq \nu_1 > 1 + \delta$  uniformly along the flow, and so

$$\begin{aligned} \int_S (u - a)^2 d\rho &= \sum_k \int_S |u_k|^2 d\rho = \sum_k \lambda_k^{-1} \int_S |\nabla u_k|^2 d\rho \\ &\leq \int_S |\nabla u_0|^2 d\rho + \sum_{k=1}^\infty (1 + \delta)^{-1} \int_S |\nabla u_k|^2 \\ &= \int_S |\nabla u_0|^2 + (1 + \delta)^{-1} \int_S |V|^2 d\rho \end{aligned}$$

We claim that  $\pi(\nabla u) = \nabla u_0$  as sections of  $\mathcal{E}^{1,0}$ . Assuming this for the moment, we obtain from (24)

$$\int_S |\nabla u_0|^2 d\rho \leq \frac{e^{-\inf u}}{Vol(S)} \langle V, V \rangle_{L^2(\mathcal{E}^{1,0}, d\mu)} \leq e^{osc(u)} \int_S |V|^2 d\rho.$$

It follows that

$$\int_S (u - a)^2 d\rho \leq \int_S |\nabla u|^2 d\rho - \frac{\delta}{1 + \delta} \int_S |V|^2 d\rho \leq \left(1 - \frac{\delta}{(1 + \delta)(1 + e^{\text{osc}(u)})}\right) \int_S |\nabla u|^2 d\rho.$$

From this, one easily shows that equation (23) holds. It suffices to establish the claim. In fact, we shall prove something more general. Suppose that  $\psi$  is an eigenfunction of  $L$  with eigenvalue  $\nu > 1$ . By elliptic regularity,  $\psi \in C_B^\infty$ . Denote by  $\nabla\psi$  the global section of  $\mathcal{E}^{1,0}$  induced by  $\bar{\partial}_\mathcal{E}\psi$ ; we claim that  $\nabla\psi \perp H^0(\mathcal{E}^{1,0})$  in  $L_B^2(d\rho)$ . To see this, let  $V \in H^0(\mathcal{E}^{1,0})$ , and compute

$$\begin{aligned} \langle V, \nabla\psi \rangle_{L^2(d\rho)} &= \int_S V^i \partial_i \bar{\psi} e^{-u} d\mu \\ &= \nu^{-1} \int_S V^i (g^T)^{i\bar{k}} (-\nabla_i^T \partial_l \bar{\psi} + \partial_i \partial_{\bar{k}} u \partial_l \bar{\psi} + \partial_{\bar{k}} u \nabla_i^T \partial_l \bar{\psi}) e^{-u} d\mu \\ &= \nu^{-1} \int_S V^i (g^T)^{i\bar{k}} (-\nabla_l^T \nabla_{\bar{k}}^T \partial_i \bar{\psi} + (-\text{Ric}_{\bar{k}i}^T + g_{\bar{k}i}^T) \partial_l \bar{\psi} + \partial_{\bar{k}} u \nabla_i^T \partial_l \bar{\psi}) e^{-u} d\mu \\ &= \nu^{-1} \int_S V^i \partial_i \bar{\psi} d\rho, \end{aligned}$$

where the last line follows by commuting covariant derivatives and integrating by parts. Since  $\nu > 1$ , we obtain the claim.  $\square$

*Proof of Proposition 9.1.* Let  $\nu$  denote the first eigenvalue of  $L$  strictly larger than 1. In light of Proposition 9.3, it suffices to show that when condition (T) holds there is a  $\delta > 0$  such that  $\nu > 1 + \delta$  along the flow. Let  $\lambda, \tilde{\lambda}$  be the smallest positive eigenvalues of the  $\bar{\partial}_\mathcal{E}$  operator acting on  $L^2(\mathcal{E}^{1,0}, d\mu)$  and  $L^2(\mathcal{E}^{1,0}, d\rho)$  respectively. That is, we have

$$\begin{aligned} \int_S |\bar{\nabla} V|^2 d\mu &\geq \lambda \int_S |V|^2 d\mu, \quad \text{for all } V \perp H^0(\mathcal{E}^{1,0}) \text{ in } L^2(d\mu) \\ \int_S |\bar{\nabla} V|^2 d\rho &\geq \tilde{\lambda} \int_S |V|^2 d\rho, \quad \text{for all } V \perp H^0(\mathcal{E}^{1,0}) \text{ in } L^2(d\rho) \end{aligned}$$

One can easily check that  $e^{-\text{osc}(u)} \lambda \leq \tilde{\lambda} \leq e^{\text{osc}(u)} \lambda$ , see for example [29, 42]. We now show that  $\nu > 1 + e^{-\text{osc}(u)} \lambda$ , which suffices to establish Proposition 9.1. Let  $\psi \in L_B^2$  be an eigenfunction of  $L$  with eigenvalue  $\lambda$ , and let  $\nabla\psi$  denote the section of  $\mathcal{E}^{1,0}$  induced by  $\bar{\partial}_\mathcal{E}\psi$ . From the proof of Proposition 9.3, we know that  $\nabla\psi \perp H^0(\mathcal{E}^{1,0})$  in  $L_B^2(d\rho)$ . We have (cf. equation (29) in [10])

$$(\nu - 1) \int_S |\nabla\psi|^2 d\rho = \int_S |\bar{\partial}_\mathcal{E} \nabla\psi|^2 d\rho \geq \tilde{\lambda} \int_S |\nabla\psi|^2 d\rho.$$

It follows that  $\nu > 1 + e^{-\text{osc}(u)} \lambda$ . By the uniform  $C^0$  bound for  $u$  in Theorem 3.1 we see that if condition (T) holds, then there is  $\delta > 0$  such that  $\nu > 1 + \delta$  uniformly along the flow.  $\square$

The following general lemma gives a condition under which the Sasaki-Ricci flow converges; in light of Proposition 9.1, it finishes the proof of Theorem 1.2.

**Lemma 9.4.** *Assume that the transverse scalar curvature  $R^T(t)$  along the Sasaki-Ricci flow satisfies*

$$\int_0^\infty \|R^T(t) - (2n + 2)n\|_{C^0} dt < \infty.$$

*Then the metrics  $g(t)$  converge exponentially fast to a Sasaki-Einstein metric.*

*Proof.* The Sasaki potential  $\phi(t)$ , satisfies equation (7), with  $F = u(0)$ , and  $\phi(0) = c_0$ , where  $c_0$  is defined by (10). For this particular choice of initial condition, Theorem 3.1 implies that  $\|\dot{\phi}\|_{C^0} \leq C$  uniformly along the flow. Now,

$$\frac{\partial}{\partial t} \log \frac{(\frac{1}{2} d\eta)^n \wedge \eta_0}{(\frac{1}{2} d\eta_0)^n \wedge \eta_0} = -(R^T - (2n + 2)n),$$

and so our assumption implies

$$\left| \log \frac{(\frac{1}{2}d\eta)^n \wedge \eta_0}{(\frac{1}{2}d\eta_0)^n \wedge \eta_0} \right| = \left| \int_0^t (R^T - (2n+2)n)dt \right| \leq \int_0^\infty \|R^T - (2n+2)n\|_{C^0} dt < \infty.$$

Rearranging equation (7) as an equation for  $\phi$ , and using the uniform bound for  $\dot{\phi}$  we obtain that  $\phi$  is uniformly bounded. By Proposition 3.3,  $\|\phi\|_{C^k}$  is uniformly bounded for each  $k \in \mathbb{N}$ , where the  $C^k$  norm is with respect to the initial metric  $g^T(0)$ . The uniform bounds on  $\phi$  imply that the metrics  $g^T(t)$  are uniformly equivalent and uniformly bounded in  $C^\infty$ ; in particular,  $Rm^T$  is uniformly bounded. It follows that there exists a subsequence of times  $t_m \rightarrow \infty$  with  $\phi(t_m)$  converging in  $C^\infty$  to smooth basic function  $\phi(\infty)$ . By uniform equivalence we have

$$|\square_{B,g(t_m)} u(t_m)| \leq C |\square_{B,g(t)} u(t_m)| \leq C |R^T(t_m) - (2n+2)n|_{C^0} \rightarrow 0.$$

Thus,  $\phi(\infty)$  is a potential for a transversely Kähler-Einstein metric. Let  $\lambda_t$  be the smallest positive eigenvalue of  $\square_\mathcal{E}$  acting on smooth global sections of  $\mathcal{E}^{1,0}$ . We claim that  $\lambda_t \geq \lambda > 0$ . If this were not the case, then there is a further subsequence (not relabeled) such that  $g(t_m)$  converges in  $C^\infty$  to a Sasaki metric  $\tilde{g}$ , and  $\lambda_{t_m} \rightarrow 0$ . We can now apply the arguments in the proof of Theorem 7.1 in the special case that the Reeb field  $\xi$  and the transverse complex structure  $\Phi$  are fixed, and  $\eta(t) = \eta_0 + 2d_B^c \phi(t)$ . In particular, by Proposition 5.3 the dimension of the space of global holomorphic sections of  $\mathcal{E}^{1,0}$  is constant. We then obtain  $0 = \lim_{m \rightarrow \infty} \lambda_{t_m} = \lambda(\tilde{g}) > 0$ , which is a contradiction. Proposition 5.1 implies that the Mabuchi K-energy is bounded below and so by Lemma 9.1 we obtain the exponential decay to zero of  $Y(t) = \|\nabla u\|_{L^2}^2$ . We claim that this implies the exponential decay to zero of  $\|\nabla u\|_{(s)}$  for any Sobolev norm  $\|\cdot\|_{(s)}$ . This follows essentially from the computations in §6. For example, rearranging equation (18), we get

$$(25) \quad (n+1)Y(t) - \int_S |\partial_B u|^2 R^T d\mu - Y(t) = \int_S |\bar{\nabla} \nabla u|^2 d\mu + \int_S |\nabla \nabla u|^2 d\mu.$$

Thus, the uniform bound for  $R^T$  and the exponential decay of  $Y$  yields the exponential decay of the right hand side of equation (25). We then proceed inductively, using the functions  $Y_{r,s}(t)$  as defined in §6 and employing the aforementioned uniform curvature bounds (cf. equation (19)). Since the transverse Ricci potential  $u$  is basic, and the metrics  $g^T(t)$  are uniformly equivalent, the Sobolev imbedding theorem yields the exponential decay to zero of  $\|u\|_{C^k}$  for any  $k$ , and hence  $\|\dot{g}_{\bar{k}j}^T\|_{C^k} = \|R_{\bar{k}j}^T - (2n+2)g_{\bar{k}j}^T\|_{C^k}$  decays exponentially to zero for any  $k$ .  $\square$

*Proof of Theorem 1.2.* Part (i) follows from Proposition 9.1, and Lemma 9.4. Part (ii) follows from the argument in the proof of Lemma 9.4. Part (iii) follows from part (ii) and part (i).  $\square$

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